$d_1 > d_3 > d_4 > d_2$, this justifies our designating x_1 as the largest root above, with $x_1 > x_3 > x_4 > x_2$. As a result, we require $x_1 + x_2 = d_1/d_2 + d_2/d_1 = 48 + 10\sqrt{21}$ and $x_3 + x_4 = d_3/d_4 + d_4/d_3 = 48 - 10\sqrt{21}$. Then rationalizing produces

$$x_1 + x_2 = \frac{2(a^2 - b^2 + c + 2a\sqrt{c})}{(a-b)^2 - c} = 48 + 10\sqrt{21},$$

so we set $a^2 - b^2 + c = 24[(a-b)^2 - c]$ and $2a = 5[(a-b)^2 - c]$. Letting $c = 21$, we obtain $48ab - 23a^2 = 5(5b^2 - 105)$ and $2a + 10ab - 5a^2 = 5b^2 - 105$. Thus $10a + 2ab - 2a^2 = 0$, so $a - b = 5$, which yields $a = 10$ and $b = 5$. Similarly, we note that $(a, b, c) = (10, 5, 21)$ produces $x_3 + x_4 = 48 - 10\sqrt{21}$ as needed.

Finally, we observe that since there is a unique real number $x > 1$ with $x + 1/x = 48 + 10\sqrt{21}$, we may conclude

$$x_1 = 24 + 5\sqrt{21} + 4\sqrt{35} + 6\sqrt{15} = \frac{\sqrt{10} + \sqrt{5 + \sqrt{21}}}{\sqrt{10} - \sqrt{5 + \sqrt{21}}}.$$

Similarly, we have the corresponding results for x_2, x_3, and x_4.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Neculai Stanciu, "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN, and the proposer.

- **5351:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Let x, y, z be positive real numbers. Show that

$$\frac{xy}{x^3 + y^3 + xyz} + \frac{yz}{y^3 + z^3 + xyz} + \frac{zx}{z^3 + x^3 + xyz} \le \frac{3}{x + y + z}.$$

Solution 1 by Ed Gray, Highland Beach, FL

Divide the numerator and denominator of the first term on the left side of the inequality by xy, and the numerator and denominator of the second term by yz and similarly the third term by zx. Thus, the left hand side becomes

$$\frac{1}{\dfrac{x^3 + y^3}{xy} + z} + \frac{1}{\dfrac{y^3 + z^3}{yz} + x} + \frac{1}{\dfrac{z^3 + x^3}{zx} + y}.$$

$$\begin{aligned} \frac{x^3 + y^3}{xy} + z &= \frac{(x+y)(x^2 - xy + y^2)}{xy} + z \\ &= (x+y)\left(\frac{x^2}{xy} - 1 + \frac{y^2}{xy}\right) + z \end{aligned}$$

$$= \quad (x+y)\left(\frac{x}{y}-1+\frac{y}{x}\right)+z$$

But $\dfrac{x}{y}+\dfrac{y}{z}-1 \geq 1$, so $\dfrac{x^3+y^3}{xy}+z \geq (x+y+z)$, and $\dfrac{1}{\dfrac{x^3+y^3}{xy}+z} \leq \dfrac{1}{x+y+z}$.

Each of the other two terms are handled in precisely the same manner, so, to avoid repetition,

$$\frac{1}{\dfrac{x^3+y^3}{xy}+z}+\frac{1}{\dfrac{y^3+z^3}{yz}+x}+\frac{1}{\dfrac{z^3+x^3}{zx}+y} \leq \frac{1}{x+y+z}+\frac{1}{y+z+x}+\frac{1}{z+x+y}=\frac{3}{x+y+z}.$$

Note that equality holds if, and only if, $x=y=z$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We have

$$\frac{xy}{x^3+y^3+xyz}+\frac{yz}{y^3+z^3+xyz}+\frac{zx}{z^3+x^3+xyz}$$

$$= \quad \frac{1}{x+y+z+\frac{(x+y)(x-y)^2}{xy}}+\frac{1}{x+y+z+\frac{(y+z)(y-z)^2}{yz}}+\frac{1}{x+y+z+\frac{(z+x)(z-x)^2}{zx}}$$

$$\leq \quad \frac{1}{x+y+z}+\frac{1}{x+y+z}+\frac{1}{x+y+z}$$

$$= \quad \frac{3}{x+y+z}, \text{ as required.}$$

Solution 3 by Arkady Alt, San Jose, CA

Since $x^3+y^3 \geq xy\,(x+y) \iff x^3+y^3-xy\,(x+y)=(x+y)\,(x-y)^2 \geq 0$ then

$$\sum_{cyc}\frac{xy}{x^3+y^3+xyz} \leq \sum_{cyc}\frac{xy}{xy\,(x+y)+xyz}=\sum_{cyc}\frac{1}{x+y+z}=\frac{3}{x+y+z}.$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Nikos Kalapodis, Patras, Greece; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria Spain; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănesti, Romania; and the proposers.